



# On minimax and Pareto optimal security payoffs in multicriteria games



Justo Puerto<sup>a,\*</sup>, Federico Perea<sup>b</sup>

<sup>a</sup> *IMUS, Universidad de Sevilla, Spain*

<sup>b</sup> *Instituto Tecnológico de Informática, Universitat Politècnica de València, Spain*

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## ABSTRACT

In this paper, we characterize minimax and Pareto-optimal security payoff vectors for general multicriteria zero-sum matrix games, using properties similar to the ones that have been used in the single criterion case. Our results show that these two solution concepts are rather similar, since they can be characterized with nearly the same sets of properties. Their main difference is the form of consistency that each solution concept satisfies. We also prove that both solution concepts can transform into each other, in their corresponding domains.

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## 1. Introduction

The standard two-person zero-sum game was introduced by von Neumann [14]. Ever since then, extensive research has been devoted to these games. Even nowadays the interest on these games has not decreased, see e.g. [13], who analyze how players' information affect their payoffs. We here elaborate on the Multicriteria Zero-Sum Matrix (MZSM) games, which allow us to handle simultaneous confrontation of two rational agents in several scenarios. The first game theoretical characterizations of minimax values in MZSM games is due to Shapley [20]. In its introductory note, F.D. Rigby remarks the importance of these games: “*The topic of games with vector payoffs is one which could be expected to attract attention on the basis of its intrinsic interest*”. Nevertheless, the development of the theory of multicriteria games has not been as successful as expected. There is a number of references in the literature as we will see along the paper, although its intrinsic difficulty, mainly due to the lack of total orderings among players' payoffs, has diminished the interest of researchers. In spite of that, the goal of further developing the analysis of multicriteria games should not be forgotten. In fact, each competitive situation that can be modeled as a scalar zero-sum game has its counterpart as a multicriteria zero-sum game, when more than one scenario has to be compared

\* Corresponding author.

E-mail addresses: [puerto@us.es](mailto:puerto@us.es) (J. Puerto), [perea@eio.upv.es](mailto:perea@eio.upv.es) (F. Perea).

simultaneously. Moreover, different scenarios do not need to have the same set (number) of strategies, which makes the analysis even more challenging. Thus, conflicting interests appear not only between different decision-makers but also within each individual, due to the different criteria they may have. For instance, the production policies of two firms which are competing for a market can be seen as a scalar game. However, when they compete simultaneously in several markets and the returns in each one of them cannot be aggregated, the multicriteria approach naturally leads to a multicriteria game. The main criticism made to this approach is its difficulty to be applied, because in most cases the solutions (values) are not unique. Therefore, new solution concepts have been proposed and compared with the existing ones.

When dealing with scalar zero sum matrix games, a natural question one wants to answer is the following: “If I play a strategy, what is the worst (to me) that the other player may do? To anticipate this, I will play the strategy that makes my worst case scenario as pleasant as possible.” This question leads to a well accepted solution concept for scalar zero sum matrix games: the minimax solution, also known as the *value of the game*. The first characterization in terms of axioms of the value of a zero-sum matrix game is due to Vilkas [22]. Later, Tijs [21] addressed the same problem, whereas more recently Norde and Voorneveld [17] and Carpenete et al. [4] provided different axiomatizations for the value of standard zero-sum matrix games. Hart et al. [12] give a Bayesian foundation of minimax values. In a natural extension to more than one criterion, players have to play the same strategy in all the different scenarios.

One of the most attractive alternatives to minimax payoff vectors is the concept of Pareto-optimal security level vectors (POSLV). Pareto-optimal security strategies (POSS) were introduced by Ghose and Prasad [11] as a solution concept in multicriteria zero-sum matrix games, extending the idea of security level strategies to more than one criterion, where players are allowed to play different strategies in each component. Ghose [10] characterized this solution concept as minimax strategies in a serial weighted zero-sum game, whereas Voorneveld [23] gave an alternative characterization of these strategies as minimax strategies of an *amalgamated* game (see [1] for the concept of amalgamation of games). Alternatively, Fernández and Puerto [6] provided a way to jointly determine POSS and their corresponding set of payoffs by solving a certain multiobjective linear program (see also [5,7,8]).

Although there exist axiomatic foundations for the minimax value of one-criteria zero-sum games, there is no characterization for the set of minimax values in the multicriteria version. On the other hand, there are several axiomatizations of Nash equilibrium strategies (see e.g. Peleg and Tijs [19], Peleg et al. [18] and Norde et al. [16]). Some of these carry over to multicriteria games, as shown in Borm et al. [2] and Voorneveld et al. [24]. Nevertheless, these are axiomatizations for strategy sets rather than for payoffs, they use different sets of axioms, and they do not clearly show the relationship between the two solutions.

In standard single criterion games, minimax and optimal security payoffs coincide. Nevertheless, whenever we have multiple criteria, which in turns may imply to have a different set (and number) of strategies in each component game, these two concepts differ. This fact raises the question of which are the common roots and which are the main differences between these two solution concepts. We will answer this question by providing characterizations that show the common properties and the differences between them.

In this paper we characterize extensions of minimax and Pareto-optimal security payoffs to general multicriteria zero-sum matrix games. Our approach uses classical properties in game theory and decision theory (objectivity, column dominance, column elimination, row dominance, row elimination and consistency). The contribution of this paper is twofold:

- we characterize solution sets rather than single value solutions which makes the analysis more involved. In this regard, we are interested in finding the largest set of payoffs compatible with the axioms proposed. This property is rather important when dealing with set-valued functions (correspondences) since it ensures that this is the largest object (solution concept) satisfying the required game theoretic properties. This approach is not new and has been already used among others by Gerard-Varet and Zamir [9] for characterizing the ‘*Reasonable set of outcomes*’ and Calleja et al. [3] for the ‘*Aggregate-monotonic core*’.

- we use a new consistency property that deals with the persistence of any solution payoff of a multicriteria game, with any given dimension on the space of criteria, on some lower dimension multicriteria game. The difference between this new consistency property and the traditional one, is the way in which solutions for a game with  $k$  criteria transform into solutions for a game with  $k - 1$  criteria. Extended minimax payoff vectors, in a multicriteria game with  $k$ -criteria, can be converted to extended minimax payoff vectors in a new  $(k - 1)$ -criteria game, that makes a convex combination of two of the original payoff matrices. However, Pareto-optimal security payoff vectors become solutions of a game with  $k - 1$  criteria, but with an amalgamation of strategies from two of the previous matrices. This difference is crucial and distinguishes the two solution concepts.

The rest of the paper is organized as follows. In Section 2 we present the basic definitions of the multicriteria games, in which we will define the two solution concepts that are the aim of this paper. In Section 3 we formally define the minimax solution concept, and characterize it using a number of axioms. Section 4 is devoted to the definition and characterization of the set of Pareto-optimal security level vectors. Section 5 is devoted to analyzing the pairwise logical independence of the axioms used in the characterizations presented. Next, in Section 6, we analyze the relationships between the two solution concepts. The paper ends with some conclusions, drawn from the results in the paper.

## 2. Multicriteria games

Let us begin by recalling the concept of scalar two-person zero-sum games.

**Definition 1.** A scalar two-person zero sum game is characterized by a payoff matrix  $A \in \mathbb{R}^{m \times n}$ , such that if player I (the row player, who wants to minimize payoffs) plays his strategy  $i$  and player II (the column player, who wants to maximize payoffs) plays his strategy  $j$ , then the row player's payoff is  $a_{ij}$  and the column player's is  $-a_{ij}$ , for  $i = 1, \dots, m, j = 1, \dots, n$ . The set of pure strategies of player I consists of  $m$  strategies, indexed by  $i$ , and the set of pure strategies of player II consists of  $n$  strategies, indexed by  $j$ . Furthermore, the set of (mixed) strategies for player I is

$$S_m := \{s \in \mathbb{R}^m : \sum_{i=1}^m s_i = 1; s_i \geq 0, \quad i = 1, \dots, m\},$$

whereas the set of (mixed) strategies for player II is  $S_n$ . In the sequel, at times we will make use of the set  $S_m^>$ , defined as the vectors in  $S_m$  with all its components strictly positive.

Note that if player I plays  $x \in S_m$  and player II plays  $y \in S_n$ , the payoff of player I is  $x^t A y \in \mathbb{R}$ , whereas player II gets the opposite. The value of the game  $A$  is defined as  $val(A) := \max_{y \in S_n} \min_{x \in S_m} x^t A y = \min_{x \in S_m} \max_{y \in S_n} x^t A y$ , which always exists (see [14]).

The traditional multicriteria approach assumes that the payoffs of the players are vectors instead of scalars. The next definition recalls this class of games, which we here denote as  $\mathcal{D}_1$ .

**Definition 2.** Consider an array of matrices  $A = (A(1), \dots, A(k)) \in \mathbb{R}^{m \times n \times k}$ , with  $A(\ell) \in \mathbb{R}^{m \times n} \forall \ell = 1, \dots, k$ . If player I plays  $x \in S_m$  and player II plays  $y \in S_n$ , the expected payoff for player I is  $x^t A y = (x^t A(1)y, \dots, x^t A(k)y) \in \mathbb{R}^k$ , and player II gets the opposite. In summary,  $\mathcal{D}_1 = \cup_{n,m,k \in \mathbb{N}} \mathbb{R}^{m \times n \times k}$ .

Note that a game in  $\mathcal{D}_1$  is uniquely characterized by its payoff matrix  $A$ , since the strategy sets only depend on the dimensions of this matrix.

Note as well that, in games in  $\mathcal{D}_1$ , each player plays the same strategy in the  $k$  payoff matrices. The following class of games, denoted by  $\mathcal{D}_2$ , allows the column player to play different strategies in the different scenarios.

**Definition 3.** Consider an array of matrices  $A = (A(1), \dots, A(k))$  with  $A(\ell) \in \mathbb{R}^{m \times n_\ell}$  for  $\ell = 1, \dots, k$  (note that there could be a different number of available strategies for the second player in the different payoff matrices). Let  $G_k = \bigcup_{m, n_1, \dots, n_k \in \mathbb{N}} \mathbb{R}^{m \times n_1} \times \dots \times \mathbb{R}^{m \times n_k}$  be the set of all such  $k$ -criteria matrices. Define the strategy spaces for players I and II as  $S_m$  and  $S_{(n_1, \dots, n_k)}$ , where

$$S_{(n_1, \dots, n_k)} := \{y = (y(1), \dots, y(k)) \in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_k} : \sum_{j=1}^{n_\ell} y(\ell)_j = 1; y(\ell)_j \geq 0, j = 1, \dots, n_\ell, \ell = 1 \dots k\}.$$

If player I plays  $x \in S_m$ , and player II plays  $y \in S_{(n_1, \dots, n_k)}$ , the payoff of player I is

$$x^t A y = (x^t A(1) y(1), \dots, x^t A(k) y(k)), \tag{1}$$

and player II gets the opposite. In summary,  $\mathcal{D}_2 = \cup_{k \in \mathbb{N}} G_k$ .

Note that a game in  $\mathcal{D}_2$  is uniquely characterized by its payoff matrix  $A$ , since the strategy sets only depend on the dimensions of the matrix.

For the sake of notation we will refer to  $\mathbb{R}^{m \times n_1} \times \dots \times \mathbb{R}^{m \times n_k}$  as  $\mathbb{R}^{m \times (n_1, \dots, n_k)}$ . The following example illustrates this class of games.

**Example 1.** Let  $F_1$  and  $F_2$  be two companies that form a duopoly competing in the same sector. The payoffs of this game will be monetary benefit and public image, measured in units that are not easily quantifiable economically. Company  $F_1$  has to face one decision: whether or not to invest in advertising (strategies  $Y$  and  $N$ , respectively). Company  $F_2$  has to face two decisions: whether or not to invest in advertising (strategies  $Y$  and  $N$ , respectively), and what to do about the polluting emissions its factory produces. Three different strategies are possible in this scenario: increase, leave as it is, decrease (denoted by  $I, L, D$ ).

Both companies know that if the two of them invest in advertising or none of them does, then their monetary benefit does not increase nor decrease. On the contrary, if one of them invests and the other does not, the company investing will have one extra unit of benefit and the other company one less unit of benefit.

If  $F_2$  increases the polluting emissions and  $F_1$  invests in publicity,  $F_1$  will use this fact in its campaign, and the public image of  $F_2$  will deteriorate, and that of  $F_1$  will improve, both by two units. In case  $F_1$  would not invest in advertising, the extra emissions will be somehow found out and the public image of  $F_2$  will deteriorate, and that of  $F_1$  will improve, both by 1 unit. If  $F_2$  decreases its emissions and  $F_1$  does not invest in publicity, the public image of  $F_2$  will improve, and that of  $F_1$  will deteriorate, both by one unit. An advertising campaign of  $F_1$  will compensate this fact, and the improvement/deterioration in public image will be of 0.5 units only.

This situation can be modeled as a game in  $\mathcal{D}_2$ , with  $m = k = n_1 = 2, n_2 = 3$  and the payoff matrix  $A = (A(1), A(2))$ , where  $A(1)$  and  $A(2)$  are:

$$A = (A(1), A(2)) = \left( \left( \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \right), \left( \begin{matrix} -2 & 0 & 0.5 \\ -1 & 0 & 1 \end{matrix} \right) \right).$$

Note that, in order to be consistent with the fact that player I is the minimizer and player II is the maximizer, the payoffs described before have been multiplied times  $-1$  when building matrix  $A$ . To illustrate, two possible such strategies are:

1.  $x = (1, 0)$  and  $y = ((0, 1), (0, 1, 0))$  are two pure strategies, that yield a payoff for player I equal to  $(0, 0)$  (player II gets the opposite).
2.  $x = (0.5, 0.5)$  and  $y = ((0.25, 0.75), (0.5, 0, 0.5))$  are two mixed strategies, that yield a payoff for player I equal to  $(-0.25, -0.375)$ .

Both for  $\mathcal{D}_1$  and  $\mathcal{D}_2$ , the scalar zero-sum game defined by the matrix  $A(\ell)$ ,  $\ell = 1 \dots k$ , will be called the  $\ell$ -component game or  $\ell$ -scenario of the corresponding multicriteria game  $A = (A(1), \dots, A(k))$ .

In the sequel, the transpose operator  $^t$  will be omitted when its use is clear. Before finishing this section, we introduce some notation that will be used in the rest of the paper.

- $B^i$  refers to the  $i$ -th row of  $B$ , and  $B^j$  refers to the  $j$ -th column of  $B$ , for any matrix  $B \in \mathbb{R}^{n \times m}$ ,  $n, m \in \mathbb{N}$ .
- Given two vectors  $a, b \in \mathbb{R}^k$ ,  $a \leq b$  if and only if  $a_\ell \leq b_\ell$  for  $\ell = 1, \dots, k$ , and  $a \lesssim b$  if  $a \leq b$  and  $a \neq b$ .
- $MIN$  ( $MAX$ ) stands for the set of minimal (maximal) elements with respect to the componentwise order of  $\mathbb{R}^k$ . So, for any  $S \in \mathbb{R}^k$ ,  $MIN(S) = \{s \in S : \text{there is no } t \in S - \{s\} : t_i \leq s_i \ \forall i = 1, \dots, k\}$  ( $MAX$  is defined analogously).

### 3. Minimax in $\mathcal{D}_1$

The multicriteria extension of the concept of minimax payoff loses some of the interesting properties shown in the scalar case: uniqueness and coincidence with security payoffs. In spite of that, it is still possible to establish the existence of such strategies under rather standard hypothesis. For instance, if the strategy set is compact, the set of *minimax* payoff vectors is non-empty, see [15].

The rationale behind minimax strategies is that each player uses the same strategy, in all the  $k$ -component games, looking for all the non-dominated vector valued payoffs. This rationale is possible in the domain  $\mathcal{D}_1$ , and so we shall consider multicriteria games defined on this domain to analyze multicriteria minimax payoff vectors.

Therefore, define as  $w_1(x)$  the worst payoffs player I can get if he/she plays strategy  $x \in S_m$ ,

$$w_1(x) := MAX\{xA(1)y, \dots, xA(k)y, y \in S_n\}. \tag{2}$$

Now, among those strategies that give componentwise maximal payoffs in the above problems, player I will choose the strategies with the *best* payoff. Note that *best* (*worst*) for player I must be understood as finding the set of minimal (maximal) elements in the componentwise ordering of  $\mathbb{R}^k$ .

**Definition 4.** The set of *extended minimax* payoffs of the game  $A \in \mathcal{D}_1$  is given by:

$$MINMAX_k(A(1), \dots, A(k)) = MIN\{\cup_{x \in S_m} w_1(x)\}, \tag{3}$$

where  $k$  is the number of different criteria. Similarly, one can define maximin payoffs.

According to the above expression, the extended minimax payoff vectors are the non-dominated vectors obtained from the solutions to all the vector valued maximization problems (2), for all  $x \in S_m$ . Clearly, if  $B \in \mathbb{R}^{m \times p}$  then  $MINMAX_1(B) = val(B)$ .

After defining the extended minimax payoff vectors, the concept of extended minimax strategy naturally follows.

**Definition 5.** An *extended minimax* strategy of player I is any strategy  $x \in S_m$  attaining an extended minimax payoff vector. Similarly, one can define extended *maximin* strategies of player II.

The first result about extended minimax payoff vectors in multicriteria games was given by Shapley [20], who provides a simple way for finding them by solving zero-sum scalar games with payoff matrix  $A(\alpha) = \sum_{\ell=1}^k \alpha_{\ell}A(\ell)$ , a positive linear combination of the matrices  $A(\ell)$ ,  $\ell = 1, \dots, k$ .

**Theorem 1** (Adapted from [20]). *Let  $z^*$  be an extended minimax value for the multicriteria game with matrix  $A = (A(1), \dots, A(k))$ . Then, there exists  $\alpha \in S_k^>$  such that  $\sum_{\ell=1}^k \alpha_{\ell}z_{\ell}^* = \text{val}(\sum_{\ell=1}^k \alpha_{\ell}A(\ell))$ .*

*Conversely, let  $z^*(\alpha)$  be the minimax value of  $(\sum_{\ell=1}^k \alpha_{\ell}A(\ell))$ , then there exists an extended minimax payoff vector  $z^* \in \text{MINMAX}_k(A(1), \dots, A(k))$ , satisfying  $\sum_{\ell=1}^k \alpha_{\ell}z_{\ell}^* = z^*(\alpha)$ .*

### 3.1. A characterization of the set of extended minimax payoff vectors

We begin this section by introducing the axioms that will characterize the set of minimax payoff vectors. Let  $\{f_k\}_{k \geq 1}$  be a family of point-to-set maps (correspondences) defined as

$$f_k : \cup_{n,m \in \mathbb{N}} \mathbb{R}^{m \times n \times k} \longrightarrow 2^{\mathbb{R}^k}$$

$$A = (A(1), \dots, A(k)) \longrightarrow f_k(A).$$

The axioms needed are:

- A0 **Objectivity.** For any  $z \in \mathbb{R}^k$ ,  $f_k(z) = z$ .
- A1 **Monotonicity.** For any  $A, \bar{A} \in \mathbb{R}^{m \times n \times k}$  such that  $\bar{A} \leq A$ ,  $f_k(\bar{A}) \subseteq f_k(A) + \mathbb{R}_-^k$ .
- A2 **Column dominance.** Let  $A_{c(\ell)}$  be the matrix that results from  $A$  after adding to the matrix  $A(\ell)$  a new column which is dominated by a convex combination of its columns. Then  $f_k(A_{c(\ell)}) = f_k(A)$ .
- A3 **Column elimination.** Let  $A_{-c(\ell)}$  be the matrix that results from  $A$  after removing column  $c(\ell)$  from the matrix  $A(\ell)$ . Then  $f_k(A_{-c(\ell)}) \subseteq f_k(A) + \mathbb{R}_-^k$ .
- A4 **Row dominance.** Let  $A_r$  be the matrix that results from  $A$  after adding a new row which is dominated by a convex combination of its rows. Then  $f_k(A_r) = f_k(A)$ .
- A5 **Row elimination.** Let  $A_{-r}$  be the matrix that results from  $A$  after removing row  $r$ . Then  $f_k(A_{-r}) \subseteq f_k(A) + \mathbb{R}_+^k$ .
- A6 **Consistency.** For any  $k \geq 2$ , if  $z \in f_k(A)$  then there exists  $0 < \alpha < 1$ , such that

$$(\alpha z_1 + (1 - \alpha)z_2, z_3, \dots, z_k) \in f_{k-1}(M(\alpha A(1), (1 - \alpha)A(2)), A(3), \dots, A(k)),$$

where  $M(\alpha A(1), (1 - \alpha)A(2))$  is a matrix with  $m$  rows, labeled  $i = 1, \dots, m$  and  $n_1 \times n_2$  columns, labeled  $c = (c(1), c(2))$  with  $c(\ell) \in \{1, \dots, n_{\ell}\}$  for each  $\ell = 1, 2$ . The entry in row  $i$  and column  $c = (c(1), c(2))$  of  $M(\alpha A(1), (1 - \alpha)A(2))$  equals  $\alpha A(1)_{ic(1)} + (1 - \alpha)A(2)_{ic(2)}$ , see [23].

- A7 **Linear consistency.** For any  $k \geq 2$  and  $A$  such that  $n_{\ell} = n \forall \ell$ , if  $z \in f_k(A)$  then there exists  $0 < \alpha < 1$  such that  $(\alpha z_1 + (1 - \alpha)z_2, z_3, \dots, z_k) \in f_{k-1}((\alpha A(1) + (1 - \alpha)A(2)), A(3), \dots, A(k))$ .

Objectivity establishes the evaluation in a trivial situation where the game has  $k$  criteria and both players have exactly one action available. Monotonicity states that the set of solution payoff vectors should not decrease, in the componentwise order of  $\mathbb{R}^k$ , when all the payoff matrices weakly increase. Column (row) dominance states that the set of solution payoff vectors should not change if player II (I) can no longer choose an action, in some of the component games, which is worse for him/her than a combination of some other actions. Column elimination states that the set of solution payoff vectors can not increase their values, in the componentwise order of  $\mathbb{R}^k$ , when some action of player II in some of the component games is removed. Row elimination states that, when removing an action of player I, the new set of solution payoff vectors must be dominated by the original one. Consistency states that any solution outcome of a game

with a given dimension, in the criteria space, can be converted into a solution outcome of a new game with lower dimension of an amalgamated game ‘à la’ Borm et al. [1]. Linear consistency states that any solution outcome provided by this correspondence, with a given dimension in the criteria space, can be converted into a solution outcome of this correspondence with lower dimension, by considering a convex combination of two of the original criteria.

The next result is a characterization of the set of extended minimax payoff vectors of any general multicriteria zero-sum game.

**Theorem 2.** *The set of extended minimax payoff vectors  $\text{MINMAX}_k$  is the largest (w.r.t. inclusion) map on  $\mathcal{D}_1$ , the set of multicriteria zero sum games, that satisfies objectivity, monotonicity, column dominance for  $k = 1$ , row dominance and linear consistency.*

**Proof.** First of all, we check that  $\text{MINMAX}_k$  satisfies the properties.

A0 – Objectivity: It is clear that  $\text{MINMAX}_k$  satisfies A0.

A1 – Monotonicity: Since  $x \geq 0$ , then  $xA(\ell) \geq x\bar{A}(\ell)$  for all  $\ell = 1, \dots, k$ . Hence, for all  $y \in S_n$ ,  $(xA(1)y, \dots, xA(k)y) \geq (x\bar{A}(1)y, \dots, x\bar{A}(k)y)$ , and the property follows.

A2 – Column dominance for  $k = 1$ : It is clear that  $\text{MINMAX}_1$  satisfies column dominance for  $k = 1$ , since  $\text{MINMAX}_1(B) = \text{val}(B)$ , and it is known that the value function,  $\text{val}(\cdot)$ , of a matrix game satisfies this property, see [4].

A4 – Row dominance: Let  $A_r = \begin{pmatrix} A(1) & \dots & A(k) \\ b_1 & \dots & b_k \end{pmatrix}$ ,  $b_\ell = (b_\ell^1, \dots, b_\ell^{n_\ell}) \in \mathbb{R}^{n_\ell}$ ,  $\ell = 1, \dots, k$ , such that  $b_\ell = \sum_{i=1}^m \alpha_i A^i(\ell)$ , with  $\sum_{i=1}^m \alpha_i = 1$  and  $\alpha_i \geq 0$ . Take  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m, \bar{x}_{m+1}) \in S_{m+1}$ . Then, for any  $y \in S_n$

$$\bar{x} \begin{pmatrix} A(\ell) \\ b_\ell \end{pmatrix} y = \sum_{i=1}^m \sum_{j \in I_\ell} (\bar{x}_i + \alpha_i \bar{x}_{m+1}) a_{ij}(\ell) y_j = \hat{x} A(\ell) y, \quad \forall \ell = 1, \dots, k,$$

where  $\hat{x} = ((\bar{x}_1 + \alpha_1 \bar{x}_{m+1}), \dots, (\bar{x}_n + \alpha_n \bar{x}_{m+1})) \in S_m$ . Hence,  $\text{MINMAX}_k(A_r) = \text{MINMAX}_k(A)$ .

A7 – Linear Consistency: Assume A7 is not satisfied. Therefore, there exists  $z = (z_1, \dots, z_k) \in \text{MINMAX}_k(A)$  such that for all  $0 < \alpha < 1$ ,  $(\alpha z_1 + (1 - \alpha)z_2, z_3, \dots, z_k) \notin \text{MINMAX}_{k-1}([\alpha A(1) + (1 - \alpha)A(2)], A(3), \dots, A(k))$ . Hence by Theorem 1, there is no  $(\beta_1, \beta_3, \dots, \beta_k) \in S_{k-1}^>$  such that

$$\beta_1 \alpha z_1 + \beta_1 (1 - \alpha) z_2 + \beta_3 z_3 + \dots + \beta_k z_k \in \text{val}(\beta_1 \alpha A(1) + \beta_1 (1 - \alpha) A(2) + \beta_3 A(3) + \dots + \beta_k A(k)).$$

However, this contradicts that  $z$  is a minimax payoff vector, because by Theorem 1, there must exist  $(\lambda_1, \dots, \lambda_k) \in S_k^>$ , such that

$$\sum_{\ell=1}^k \lambda_\ell z_\ell = \text{val}(\lambda_1 A(1) + \dots + \lambda_k A(k)).$$

The above contradiction proves that  $\text{MINMAX}_k$  satisfies A7.

To finish the proof, it is enough to show that if  $\{g_k\}_{k \geq 1}$  is an arbitrary family of point-to-set maps that satisfy A0, A1, A2 for  $k = 1$ , A4 and A7, then for any general multicriteria game given by a matrix  $A = (A(1), \dots, A(k))$ , with  $A(\ell) \in \mathbb{R}^{m \times n}$  for all  $\ell = 1, \dots, k$ , we get  $g_k(A) \subseteq \text{MINMAX}_k(A) \forall k \geq 1$ .

For  $k = 1$ , the axioms A0, A1, A2 and A4 characterize the  $\text{val}(\cdot)$  function of a matrix game (see Carpenente et al. [4, Theorem 2]). Therefore,  $g_1(\cdot) = \text{val}(\cdot) = \text{MINMAX}_1(\cdot)$ .

Let  $z \in g_k(A), k \geq 2$ . Apply A7 ( $k - 1$ )-times to conclude that there exists  $\alpha \in \mathbb{R}^k, \sum_{\ell=1}^k \alpha_\ell = 1, \alpha_\ell > 0$ , such that

$$\sum_{\ell=1}^k \alpha_\ell z_\ell = \text{val}\left(\sum_{\ell=1}^k \alpha_\ell A(\ell)\right).$$

Then, by Theorem 1,  $z$  is an extended minimax payoff vector of the game defined by  $A$ . Hence,

$$g_k(A) \subseteq \text{MINMAX}_k(A) \quad \forall A, k \geq 1. \quad \square$$

The above theorem also implies that  $\text{MINMAX}_k$  is the largest map on  $\mathcal{D}_1$  that satisfies Linear Consistency (A7), and that coincides with the value function on standard single criterion matrix games. Another characterization of extended minimax payoff vectors using a different set of properties is possible. The rationale is to alternatively characterize the  $\text{val}(\cdot)$  function and then to apply the consistency construction. This is possible based on Carpenente et al. [4, Theorem 3].

**Theorem 3.** *The set of extended minimax payoff vectors  $\text{MINMAX}_k$  is the largest (w.r.t. inclusion) map on  $\mathcal{D}_1$  that satisfies objectivity, column dominance, row dominance, column elimination, row elimination and linear consistency.*

The proof runs similarly to that of Theorem 2, but using [4, Theorem 3] instead of [4, Theorem 2].

#### 4. POSS in $\mathcal{D}_2$

This solution concept is independent of the notion of equilibrium, so that the opponents are only taken into account to establish the security levels for one’s own payoff. Therefore, this notion does not require to play the same strategy in all the scalar  $\ell$ -component games  $A(\ell)$ , and thus is defined in the class  $\mathcal{D}_2$ .

**Definition 6.** Every strategy  $x \in S_m$  defines security levels  $v_I^\ell(x)$  as the payoffs with respect to each criterion, when II bets to maximize the criteria [10]. Hence,

$$v_I^\ell(x) = \max_{y \in S_{n_\ell}} xA(\ell)y, \ell = 1, \dots, k, \tag{4}$$

and the security levels are  $k$ -tuples denoted by

$$v_I(x) = (v_I^1(x), \dots, v_I^k(x)). \tag{5}$$

We will use the notation  $v_I(x, A)$  to specify such matrix  $A$ , whenever this is needed to avoid confusion. It must be noted that for a given strategy  $x$  for player I, the security levels  $v_I(x)$  might be obtained by different strategies of player II. In [11] the concept of Pareto optimal security strategy (POSS) is defined in  $\mathcal{D}_1$  as follows. It should be noted that the definition on  $\mathcal{D}_2$  is the same.

**Definition 7.** A strategy  $x^* \in S_m$  is a Pareto-optimal security strategy for I if and only if there is no  $x \in S_m$  such that  $v_I(x^*) \succeq v_I(x)$ . Similarly, one can define POSS for II.

The set of Pareto-optimal security level vectors is the set of payoffs that can be attained by POSS, and will be denoted by  $\text{VPOSS}_k(A(1), \dots, A(k))$ , thus

$$\text{VPOSS}_k(A(1), \dots, A(k)) = \{z \in \mathbb{R}^k : z = v_I(x) \text{ for some } x \in S_m \text{ and } \nexists x' \in S_m \text{ such that } v_I(x') \prec z\}. \tag{6}$$



The reader should observe that there are extended minimax values that cannot be obtained by Pareto-optimal security level vectors. The reason is that POSS attain inferior payoffs by allowing the column player to change their strategies in the different criteria games. This fact will be made clearer in Section 6.

The following theorem provides a way to determine all POSS and their security level vectors.

**Theorem 4.** (*[6, Theorem 3.1]*) *A strategy  $x^* \in S_m$  is a POSS and  $v^* = (v_1^*, \dots, v_k^*)$  is its security level vector if and only if  $(v^*, x^*)$  is an efficient solution to the problem*

$$\begin{aligned} &\min (v_1, \dots, v_k), \\ &\text{s.t. } xA(\ell) \leq (v_\ell, \dots, v_\ell), \ell = 1, \dots, k, \\ &\quad \sum_{i=1}^n x_i = 1, \\ &\quad x \geq 0, v \in \mathbb{R}^k. \end{aligned}$$

Equivalently, Voorneveld [23, Theorem 3.1] characterize POSS strategies as minimax strategies of particular classes of scalar games.

**Theorem 5.** (*[23, Theorem 3.1]*) *A strategy  $x^* \in S_m$  is a POSS for player I in the multicriteria matrix game  $A = (A(1), \dots, A(k))$  if and only if there exists a vector  $\alpha \in S_k^>$  such that  $x^*$  is a minimax strategy in the scalar matrix game  $M(\alpha_1 A(1), \dots, \alpha_k A(k))$  being  $M(\alpha_1 A(1), \dots, \alpha_k A(k))$  a matrix with  $m$  rows, labeled  $i = 1, \dots, m$  and  $n^k$  columns, labeled  $c = (c(1), \dots, c(k))$  with  $c(\ell) \in \{1, \dots, n\}$  for each  $\ell = 1, \dots, k$ . The entry in row  $i$  and column  $c = (c(1), \dots, c(k))$  of  $M(\alpha A)$  equals  $\sum_{\ell=1}^k \alpha_\ell A(\ell)_{ic(\ell)}$ .*

The reader may note that in the scalar case,  $VPOSS_1(B) = \text{MINMAX}_1(B) = \text{val}(B)$ , and Theorems 4 and 5 coincide.

#### 4.1. A characterization of Pareto-optimal security payoffs

In this section, we characterize the set of Pareto-optimal security payoffs defined on a general multicriteria two-person zero-sum game in  $\mathcal{D}_2$ . Thus, we try to identify a map (solution concept) that associates to any array of  $k$  matrices with the same number of rows a set of vectors in  $\mathbb{R}^k, k \geq 1$ .

Using the set of properties introduced in Section 3.1, we can obtain a characterization of the entire set of Pareto-optimal security payoff vectors as the maximal (in the inclusion sense) point-to-set map that satisfies A0, A1, A2, A4 and A6.

**Theorem 6.** *The set of Pareto-optimal security level vectors  $VPOSS_k$  is the largest (w.r.t. inclusion) map on  $\mathcal{D}_2$  that satisfies objectivity, monotonicity, column dominance, row dominance and consistency.*

**Proof.** The proof is similar to Theorem 2, but using Theorem 5 instead of Theorem 1. First of all, we check that  $VPOSS_k$  satisfies the properties of objectivity, monotonicity, column dominance, row dominance and consistency.

*A0 – Objectivity:* Clearly,  $VPOSS_k$  satisfies A0.

*A1 – Monotonicity:* The security level vectors for the strategy  $x$  with respect to  $A$  are:

$$\begin{aligned} (\max_{y \in S_{n_1}} xA(1)y, \dots, \max_{y \in S_{n_k}} xA(k)) &= (\max_{y \in \text{ext}\{S_{n_1}\}} xA(1)y, \dots, \max_{y \in \text{ext}\{S_{n_k}\}} xA(k)) \\ &= (\max\{xA(1)\}, \dots, \max\{xA(k)\}), \end{aligned}$$

where  $\max\{xA(\ell)\}$  denotes the maximum component of the vector  $xA(\ell)$ . Note that this is true, because  $\text{ext}\{S_{n_\ell}\}$  (the set of extreme points of  $S_{n_\ell}$ ) consists of the vectors whose  $i$ th component is one and the rest is zero, for  $i = 1, \dots, n_\ell$ . Analogously,

$$v_I(x, \bar{A}) = (\max\{x\bar{A}(1)\}, \dots, \max\{x\bar{A}(k)\}).$$

Now, since  $x \geq 0$ , then  $xA(\ell) \geq x\bar{A}(\ell)$ , for all  $\ell = 1, \dots, k$ . Hence,  $v_I(x, A) \geq v_I(x, \bar{A})$ , for each  $x$ , and the conclusion follows.

*A2 – Column dominance:* Let  $A_{c(\ell)}(\ell)$  be the matrix that results from adding to  $A(\ell)$  a new column  $H$ , that is dominated by a convex combination of the columns of  $A(\ell)$ , i.e.,  $A_{c(\ell)}(\ell) = (A^{-1}(\ell), \dots, A^{n_\ell}(\ell), H)$  and there exists  $\alpha_1, \dots, \alpha_{n_\ell} \geq 0$  such that  $\sum_{j=1}^{n_\ell} \alpha_j = 1$  satisfying  $\sum_{\ell=1}^k \alpha_j A^{-j}(\ell) \geq H$ . Clearly,  $A_{c(\ell)}(s) = A(s) \forall s \neq \ell$ . By construction,

$$v_I^\ell(x, A_{c(\ell)}) = \max(xA_{c(\ell)}^{-1}(\ell), \dots, xA_{c(\ell)}^{n_\ell}(\ell), xH) = \max(xA^{-1}(\ell), \dots, xA^{n_\ell}(\ell)) = v_I^\ell(x, A).$$

Then,  $v_I^s(x, A_{c(\ell)}) = v_I^s(x, A) \forall s$ , and thus  $v_I(x, A_{c(\ell)}) = v_I(x, A)$ .

*A4 – Row dominance:* Let  $A_r = \begin{pmatrix} A(1) & \dots & A(k) \\ b_1 & \dots & b_k \end{pmatrix}$ ,  $b_\ell = (b_\ell^1, \dots, b_\ell^{n_\ell}) \in \mathbb{R}^{n_\ell}$ ,  $\ell = 1, \dots, k$  such that  $b_\ell = \sum_{i=1}^m \alpha_i A^{i \cdot}(\ell)$  with  $\sum_{i=1}^m \alpha_i = 1$  and  $\alpha_i \geq 0$ . Take  $\bar{x} \in S_{m+1}$ ; then

$$v_I(\bar{x}, A_r) = (\max \bar{x}A_r(1), \dots, \max \bar{x}A_r(k)).$$

The security level of  $A_r$  in the  $\ell$ -th component is

$$\begin{aligned} v_I^\ell(\bar{x}, A_r) &= \max(\sum_{i=1}^m \bar{x}_i A^{-1}(\ell) + \bar{x}_{m+1} b_\ell^1, \dots, \sum_{i=1}^m \bar{x}_i A^{n_\ell}(\ell) + \bar{x}_{m+1} b_\ell^{n_\ell}) \\ &= \max(\sum_{i=1}^m (\bar{x}_i + \alpha_i \bar{x}_{m+1}) A^{-1}(\ell), \dots, \sum_{i=1}^m (\bar{x}_i + \alpha_i \bar{x}_{m+1}) A^{n_\ell}(\ell)) = v_I^\ell(\hat{x}, A), \end{aligned}$$

where  $\hat{x} = (\bar{x}_1 + \alpha_1 \bar{x}_{m+1}, \dots, \bar{x}_m + \alpha_m \bar{x}_{m+1}) \in S_m$ .

Then, for any  $\bar{x} \in S_{m+1}$ ,  $\exists \hat{x} \in S_m$ , such that  $v_I^\ell(\bar{x}, A_r) = v_I^\ell(\hat{x}, A_r)$ ,  $\forall \ell$ , and conversely.

*A6 – Consistency:* If A6 is not satisfied, then there exists  $z = (z_1, \dots, z_k) \in VPOSS_k(A)$  such that for all  $\alpha \in (0, 1)$ ,  $(\alpha z_1 + (1 - \alpha)z_2, z_3, \dots, z_k) \notin VPOSS_{k-1}(M[\alpha A(1), (1 - \alpha)A(2)], A(3), \dots, A(k))$ . Hence, by [23, Theorem 3.1],  $\nexists (\beta_1, \beta_3, \dots, \beta_k) \in S_{k-1}^>$  such that

$$\beta_1 \alpha z_1 + \beta_1 (1 - \alpha) z_2 + \beta_3 z_3 + \dots + \beta_k z_k = \text{val}(M[\beta_1 M[\alpha A(1), (1 - \alpha)A(2)], \beta_3 A(3), \dots, \beta_k A(k)]).$$

However, this contradicts that  $z$  is a payoff vector of a POSS, since by [23, Theorem 3.1] there must exist  $(\lambda_1, \dots, \lambda_k) \in S_k^>$ , such that

$$\sum_{\ell=1}^k \lambda_\ell z_\ell = \text{val}(M[\lambda_1 A(1), \dots, \lambda_k A(k)]).$$

Thus,  $VPOSS_k$  satisfies A6.

To finish the proof, it is enough to show that, if  $\{f_k\}_{k \geq 1}$  is an arbitrary family of point-to-set maps that satisfy A0, A1, A2, A4, and A6, then, for any general multicriteria game given by the matrix  $A = (A(\ell))_{\ell=1 \dots k}$  with  $A(\ell) \in \mathbb{R}^{m \times n_\ell}$ , we get  $f_k(A) \subseteq VPOSS_k(A)$ ,  $\forall k \geq 1$ .

Indeed, for  $k = 1$ , the axioms A0, A1, A2 and A4 characterize the  $\text{val}(\cdot)$  function of a matrix game (see Carpenete et al., [4, Theorem 2]). Therefore,  $f_1(\cdot) = \text{val}(\cdot)$ .

Let  $z \in f_k(A)$ ,  $k \geq 2$ . Apply A6 ( $k - 1$ )-times to conclude that there exists  $\alpha \in \mathbb{R}^k$  satisfying  $\sum_{\ell=1}^k \alpha_\ell = 1$ ,  $\alpha_\ell > 0$ , such that  $\sum_{\ell=1}^k \alpha_\ell z_\ell = \text{val}(M[\alpha_1 A(1), \dots, \alpha_k A(k)])$ . Then, by [23, Theorem 3.1],  $z$  is a payoff vector of a POSS of the multi-matrix  $A$ . Hence,  $f_k(A) \subseteq VPOSS_k(A) \forall A$ ,  $k \geq 1$ .  $\square$

After a careful reading of the proof, one realizes that an alternative characterization is still possible using weaker versions of properties A0–A4.

**Corollary 1.** *If the properties A0, A1, A2 and A4 are required only for  $k = 1$ , the characterization of Theorem 6 still holds.*

This corollary implies that  $VPOSS_k$  is the largest map on  $\mathcal{D}_2$  that satisfies consistency (A6), and coincides with the value function on standard single criterion matrix games. Another characterization of Pareto-optimal security level vectors, using a different set of properties, is possible by alternatively characterizing the  $val(\cdot)$  function to then apply the consistency construction. This is possible thanks to Carpen- te et al. [4, Theorem 3].

**Theorem 7.** *The set of Pareto-optimal security level vectors  $VPOSS_k$  is the largest (w.r.t. inclusion) map on  $\mathcal{D}_2$  that satisfies objectivity, column dominance, row dominance, column elimination, row elimination and consistency.*

## 5. Pairwise logical independence of the properties

This section shows that the previously presented characterizations use pairwise logically independent properties. In doing that, we will use some results from the literature and three additional evaluation maps.

First of all, we observe that since properties A0–A7 must hold for any  $k \geq 1$ , it is enough to show that there exist evaluation maps, for particular choices of  $k$ , fulfilling only some of these properties. In most cases, this is possible for  $k = 1$ . We introduce three evaluation maps  $h_0, h_1, h_2$ , defined on  $G_1 = \bigcup_{m,p \in \mathbb{N}} \mathbb{R}^{m \times p}$ , the space of the scalar zero-sum games,  $h_i : G_1 \rightarrow \mathbb{R}$  for  $i = 0, 1, 2$ , such that for any  $B \in G_1$ :

$$h_0(B) = 0, \quad (7)$$

$$h_1(B) = b_{11}, \quad (8)$$

$$h_2(B) = \min_{1 \leq i \leq \text{row}(B)} b_{i1}, \quad (9)$$

where  $\text{row}(B)$  is the number of rows of the matrix  $B$ .

**A0:** First of all, to show that objectivity (A0) is logically independent of the rest of properties, we use the null function  $h_0$ . For  $k = 1$ , [17] proves that the null function satisfies monotonicity (A1), row dominance (A4), and row elimination (A5). It is also easy to see that  $h_0$  also satisfies column elimination (A3). However,  $h_0$  does not satisfy objectivity (A0). In addition, [22] proves using its function  $f_4$  that objectivity (A0) and column dominance (A2) are independent.

**A1:** For  $k = 1$ , the independence of monotonicity (A1) from column dominance (A2) follows from [22, Theorem 4]; from column elimination (A3) easily follows using our function  $h_1$ ; and from row dominance (A4) and row elimination (A5) is proved in [17] using its function  $f_6$ .

**A2:** Function  $h_1$  satisfies column dominance (A2) but does not verify column elimination (A3) nor row elimination (A5). Besides, Theorem 4 in [22] proves that column dominance (A2) and row dominance (A4) are independent.

**A3:** Function  $h_1$  shows that column elimination (A3) and row dominance (A4) are independent. The function  $h_2$  satisfies row elimination (A5), but does not satisfy column elimination (A3). Moreover, [17] shows with its function  $f_4$  that row dominance (A4) and column elimination (A3) are independent.

**A6, A7:** Finally, to prove that objectivity (A0), monotonicity (A1), column dominance (A2), column elimination (A3), row dominance (A4) and row elimination (A5) are independent of consistency (A6) and linear consistency (A7), we use the correspondences  $\text{MINMAX}_k$  and  $VPOSS_k$ , respectively. Examples 2 and 3 show that  $\text{MINMAX}_k$  does not satisfy Consistency (A6) and  $VPOSS_k$  does not satisfy Linear Consistency (A7), respectively.

Carpen- te et al. [4] show that the minimax value satisfies axioms A0–A5. We show in the next example that this solution concept does not satisfy Consistency.

**Table 1**  
Pairwise logical independence of properties.

	A0	A1	A2	A3	A4	A5	A6	A7
A0	X	$h_0$ (7)	$f_4$ in [22]	$h_0$ (7)	$h_0$ (7)	$h_0$ (7)	MINMAX <sub>k</sub>	VPOSS <sub>k</sub>
A1		X	Th. 4 in [22]	$h_1$ (8)	$f_6$ in [17]	$f_6$ in [17]	MINMAX <sub>k</sub>	VPOSS <sub>k</sub>
A2			X	$h_1$ (8)	Th. 4 in [22]	$h_1$ (8)	MINMAX <sub>k</sub>	VPOSS <sub>k</sub>
A3				X	$h_1$ (8)	$h_2$ (9)	MINMAX <sub>k</sub>	VPOSS <sub>k</sub>
A4					X	$f_4$ in [17]	MINMAX <sub>k</sub>	VPOSS <sub>k</sub>
A5						X	MINMAX <sub>k</sub>	VPOSS <sub>k</sub>
A6							X	VPOSS <sub>k</sub>
A7								X

**Example 2.** Consider the 2-criteria game defined by the matrices  $A(1) = (1, 0)$ ,  $A(2) = (0, 1)$ . Note that player I only has one pure strategy while player II has two pure strategies in this 2-criteria game. The minimax values of this game are  $(\alpha, 1 - \alpha)$ ,  $\forall \alpha \in [0, 1]$ . The reader may note that these values do not satisfy *Consistence* (A6). Indeed,

$$M[\alpha A(1), (1 - \alpha)A(2)] = (\alpha, 1, 0, 1 - \alpha).$$

It is clear that the minimax value of the single criterion game with the above matrix is 1. Hence, it can not be a convex combination of  $(\alpha, 1 - \alpha)$ , the minimax values of the original 2-criteria game, for any  $\alpha \in (0, 1)$ .

**Example 3.** (2 continued)

Consider the game given in [Example 2](#), described by the two matrices  $A(1) = (1, 0)$ ,  $A(2) = (0, 1)$ . For this game, the unique Pareto-optimal security payoff is  $(1, 1)$ . Let us now consider the game given by the matrix:

$$\alpha A(1) + (1 - \alpha)A(2) = (\alpha, 1 - \alpha),$$

with  $1/2 < \alpha \leq 1$ . The value of this game is  $\alpha$ . Therefore, this security payoff does not satisfy the property of *linear consistency* for any  $\alpha \in (0, 1)$ , because  $\alpha$  cannot be obtained as a convex combination of  $(1, 1)$ . The reader may note that the minimax payoff  $(1, 0)$  does satisfy this property for  $\alpha \in [1/2, 1]$ .

[Table 1](#) summarizes the pairwise logical independence of properties.

### 6. Relationship between minimax and POSS in $\mathcal{D}_1$ and $\mathcal{D}_2$

In this section we show the similarities between minimax (in  $\mathcal{D}_1$ ) and POSS (in  $\mathcal{D}_2$ ) by proving that, when a game in  $\mathcal{D}_2$  is transformed into a game in  $\mathcal{D}_1$ , then the corresponding POSS transforms into minimax.

Firstly, the next results shows a transformation of a game in  $\mathcal{D}_2$  into a game in  $\mathcal{D}_1$ , and how the corresponding strategies can also be transformed, keeping the same payoffs. The reader may note that this transformation is only of theoretical interest since it is not polynomial in the input size of the game  $A$ . Indeed, it requires to construct a game with an exponential number of strategies in  $\mathcal{D}_1$  based on some form of amalgamation operation, as explained later.

**Theorem 8.** *Every game  $A \in \mathcal{D}_2$  can be transformed into a game  $\bar{A} \in \mathcal{D}_1$ . Besides, given  $(x, y)$  strategies in  $A$  for players I and II, there exist  $(\bar{x}, \bar{y})$  strategies for I and II in  $\bar{A}$  such that  $xAy = \bar{x}\bar{A}\bar{y}$ .*

**Proof.** Given is  $A = (A(1), \dots, A(k))$ , with  $A(\ell) \in \mathbb{R}^{m \times n_\ell} \forall \ell = 1, \dots, k$ , which defines a game in  $\mathcal{D}_2$ . The strategy sets for players I and II are  $S_m$  and  $S_{(n_1, \dots, n_k)}$ .

Consider  $\bar{A} = (\bar{A}(1), \dots, \bar{A}(k))$ , with  $\bar{A}(\ell) \in \mathbb{R}^{m \times \prod_{\ell} n_{\ell}}$ , which defines a game in  $\mathcal{D}_1$ , built in such a way that  $\bar{A}(\ell)_{i,(j_1, \dots, j_k)} = A(\ell)_{ij_{\ell}}$ ,  $\forall i = 1, \dots, m, j_{\ell} = 1, \dots, n_{\ell}, \ell = 1, \dots, k$ . The strategy sets for players I and II are  $S_m$  (the same as in game  $A$ ) and  $S_{\Pi} := \{y \in \mathbb{R}^{\prod_{\ell} n_{\ell}} : \sum_{(j_1, \dots, j_k)} y_{(j_1, \dots, j_k)} = 1, y \geq 0\}$ .

Given two strategies  $x = (x_1, \dots, x_m) \in S_m$  and  $y = (y(1), \dots, y(k)) \in S_{(n_1, \dots, n_k)}$ , define  $\bar{x} = x$  and  $\bar{y} : \bar{y}_{(j_1, \dots, j_k)} = \prod_{\ell} y(\ell)_{j_{\ell}}$ . Let us now prove that  $(\bar{x}, \bar{y})$  satisfy the conditions of the theorem.

- Clearly,  $\bar{x} \in S_m$ . Let us prove that  $\bar{y} \in \bar{S}_{\Pi}$ .  
 $\sum_{(j_1, \dots, j_k)} \bar{y}_{(j_1, \dots, j_k)} = \sum_{(j_1, \dots, j_k)} \prod_{\ell} y(\ell)_{j_{\ell}} = \prod_{\ell} \sum_{j_{\ell}} y(\ell)_{j_{\ell}} = 1$ .
- Now, let us check that  $xAy = \bar{x}\bar{A}\bar{y}$ . For this purpose, we need to prove that  $x A(\ell) y(\ell) = \bar{x} \bar{A}(\ell) \bar{y}$ ,  $\forall \ell = 1, \dots, k$ . Take  $\ell \in \{1, \dots, k\}$ . Because  $x A(\ell) y(\ell) = \sum_i x_i (A(\ell) y(\ell))_i$ , and  $\bar{x} \bar{A}(\ell) \bar{y} = \sum_i \bar{x}_i (\bar{A}(\ell) \bar{y})_i = \sum_i x_i (\bar{A}(\ell) \bar{y})_i$ , the proof of this statement reduces to check that  $(A(\ell) y(\ell))_i = (\bar{A}(\ell) \bar{y})_i$ ,  $\forall i = 1, \dots, m$ .  
  1.  $(A(\ell) y(\ell))_i = \sum_{j_{\ell}} A(\ell)_{i, j_{\ell}} y(\ell)_{j_{\ell}}$ .
  - 2.

$$\begin{aligned} (\bar{A}(\ell) \bar{y})_i &= \sum_{(j_1, \dots, j_k)} \bar{A}(\ell)_{i, (j_1, \dots, j_k)} \bar{y}_{(j_1, \dots, j_k)} \\ &= \sum_{(j_1, \dots, j_k)} A(\ell)_{i, j_{\ell}} \prod_{\ell'} y(\ell')_{j_{\ell'}} \\ &= \sum_{j_{\ell}} (\sum_{J \setminus j_{\ell}} A(\ell)_{i, j_{\ell}} \prod_{\ell'} y(\ell')_{j_{\ell'}}) \\ &= \sum_{j_{\ell}} A(\ell)_{i, j_{\ell}} y(\ell)_{j_{\ell}} (\sum_{J \setminus j_{\ell}} \prod_{\ell' \neq \ell} y(\ell')_{j_{\ell'}}) \\ &= \sum_{j_{\ell}} A(\ell)_{i, j_{\ell}} y(\ell)_{j_{\ell}} (\prod_{\ell' \neq \ell} \sum_{j_{\ell'}} y(\ell')_{j_{\ell'}}) \\ &= \sum_{j_{\ell}} A(\ell)_{i, j_{\ell}} y(\ell)_{j_{\ell}} = (A(\ell) y(\ell))_i, \end{aligned}$$

where  $J$  is sometimes used to denote the complete vector of indexes  $(j_1, \dots, j_k)$ .

This proves that  $(\bar{A}(\ell) \bar{y})_i = (A(\ell) y(\ell))_i \forall \ell, i$ , and therefore  $xAy = \bar{x}\bar{A}\bar{y}$ .  $\square$

Let us denote by  $\bar{\mathcal{D}}_1$  the class of games in  $\mathcal{D}_1$  that can be obtained from a game in  $\mathcal{D}_2$  as in [Theorem 8](#). The following lemma states that strategies in a game in  $\bar{\mathcal{D}}_1$  can be transformed into strategies in the corresponding game in  $\mathcal{D}_2$ , keeping the same payoffs.

**Lemma 1.** *Let  $A$  be a game in  $\mathcal{D}_2$ , and let  $\bar{A}$  be its corresponding transformation into a game in  $\bar{\mathcal{D}}_1$ . Let  $(x, \bar{y})$  be a pair of mixed strategies for players I and II in game  $\bar{A}$ . Then, there exists a pair of strategies  $(x, y)$  for the game  $A$  such that  $x\bar{A}\bar{y} = xAy$ .*

**Proof.** Let  $(x, \bar{y}) \in S_m \times S_{\Pi}$  be a pair of strategies for game  $\bar{A}$ . Build  $y = (y(1), \dots, y(k))$  so  $y(\ell)_{j_{\ell}} = \sum_{(j_1, \dots, j_k) : j_{\ell} = j} \bar{y}_{j_1, \dots, j_k}$ , for  $j = 1, \dots, n_{\ell}, \ell = 1, \dots, k$ . Clearly,  $\sum_{j_{\ell}} y(\ell)_{j_{\ell}} = 1$ , and all such components are positive, therefore  $y \in S_{(n_1, \dots, n_k)}$ .

Let us now prove that both payoffs are equal. Take one of the criteria,  $\ell$ . We have that  $x\bar{A}(\ell)\bar{y} = \sum_i \sum_J x_i \bar{A}(\ell)_{i, J} \bar{y}_J$ , and  $x A(\ell) y = \sum_i \sum_{j_{\ell}} x_i A(\ell)_{i, j_{\ell}} y_{j_{\ell}}$ . The following equation proves that both payoffs are equal:

$$\sum_J \bar{A}(\ell)_{i, J} \bar{y}_J = \sum_J A(\ell)_{i, j_{\ell}} \bar{y}_J = \sum_{j_{\ell}} A(\ell)_{i, j_{\ell}} \sum_{J \setminus \{j_{\ell}\}} \bar{y}_J = \sum_{j_{\ell}} A(\ell)_{i, j_{\ell}} y_{j_{\ell}}. \quad \square$$

**Example 4.** Applying the transformation in [Theorem 8](#) to the game in [Example 1](#), the new payoff matrix is:  $\bar{A} = (\bar{A}(1), \bar{A}(2))$ , with:

$$(\bar{A}(1), \bar{A}(2)) = \left( \left( \begin{pmatrix} 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 0.5 & -2 & 0 & 0.5 \\ -1 & 0 & 1 & -1 & 0 & 1 \end{pmatrix} \right) \right).$$

Note that this new game has two criteria, two strategies for player I, and six ( $3 \times 2$ ) strategies for player II. The strategy sets for the players are therefore  $S_2$  and  $S_6$ .

The next theorem is the main result of this section, and states that POSS strategies in  $\mathcal{D}_2$  can be transformed into minimax strategies in  $\bar{\mathcal{D}}_1$ , and vice versa.

**Theorem 9.** *Let  $A$  be a game in  $\mathcal{D}_2$ , and let  $\bar{A}$  be its corresponding transformation into a game in  $\bar{\mathcal{D}}_1$ . Then we have that  $(x^*, y^*)$  is a pair of POSS in game  $A$  if and only if  $(x^*, \bar{y}^*)$  is a pair of minimax strategies in game  $\bar{A}$ , where  $\bar{y}^* \in S_{\Pi}$  is obtained from  $y^* \in S_{(n_1, \dots, n_k)}$  as in [Theorem 8](#).*

**Proof.** Let us prove both implications.

1. Let us first prove that  $x^*$  is a POSS for I in  $A$  if and only if  $x^*$  is minimax for I in  $\bar{A}$ . For any strategy of player II, the worst payoff player I can obtain from  $A(\ell)$  is the same as the worst payoff he/she can obtain from  $\bar{A}(\ell)$  (note that the columns of  $\bar{A}(\ell)$  are the same as in  $A(\ell)$ , but repeated and in different order). And therefore the concept of POSS for I in  $A$ , and the concept of minimax for I in  $\bar{A}$  coincide.
2. Now let us prove that  $y^*$  is POSS for II in  $A$  if and only if  $\bar{y}^*$  is maximin for II in  $\bar{A}$ . Because the payoffs are the same in both games, that is, for any  $x \in S_m$ ,  $x A(\ell) y^* = x \bar{A}(\ell) \bar{y}^*$ , we have that for any strategy of I, the worst payoff player II can obtain from  $A$  is the same as the worst payoff player II can obtain for  $\bar{A}$ , and therefore the concept of POSS for II in  $A$  coincide with the concept of maximin for II in  $\bar{A}$ .  $\square$

Yet another characterization of the set of Pareto-optimal security level vectors is possible. This new characterization is a by-product of [Theorems 8 and 9](#), above.

**Corollary 2.** *The set of Pareto-optimal security level vectors  $VPOSS_k$  is the largest (w.r.t. inclusion) map on  $\bar{\mathcal{D}}_1$  that satisfies objectivity, column dominance, row dominance, column elimination, row elimination and linear consistency.*

**Proof.** By [Theorems 8 and 9](#) every  $A \in \mathcal{D}_2$  can be transformed into  $\bar{A} \in \bar{\mathcal{D}}_1$  such that  $(x^*, y^*)$  is a POSS in  $A$  with value  $x^* A y^*$  if and only if  $(x^*, \bar{y}^*)$  is a extended minimax in  $\bar{A}$  and  $x^* A y^* = x^* \bar{A} \bar{y}^*$ . Therefore,  $VPOSS$  in  $\mathcal{D}_2$  are v-minimax in  $\bar{\mathcal{D}}_1$ . Next, we can apply [Theorem 3](#) to the set  $\bar{\mathcal{D}}_1$  and the results follow.  $\square$

## 7. Conclusions

In this paper, we study minimax and POSS strategies, two well-known solution concepts already defined in the literature. Whereas for minimax strategies both players need to play the same strategy in all scenarios, POSS strategies allow the column player to play different strategies in different scenarios in order to improve his/her security levels. Two axiomatic characterizations for these solution concepts in their corresponding domains, that share all axioms but one, show that minimax and POSS strategies are very similar to each other. Comparing [Theorems 2 and 7](#), we realize that Pareto-optimal security level vectors and extended minimax payoff vectors differ only in the way in which solution payoff vectors from  $k$ -criteria games are transformed into solution payoff vectors games with  $(k - 1)$  criteria (consistency properties). The former requires to amalgamate strategies in a game with lower dimension, and the latter requires convex combinations of payoff matrices. Besides, we have also shown that POSS strategies can be transformed into minimax strategies, in their corresponding domains.

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